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Wavefunctions for the time-dependent linear oscillator and Lie point symmetries

K Andriopoulos[1] **and P G L Leach**[2]

[1] Department of Mathematics, National and Capodistrian University of Athens, Panepistimioupolis, Ilisia, 15771 Athens, Greece

[2] School of Mathematical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, South Africa

E-mail: kand@aegean.gr, leachp@ukzn.ac.za and leach@math.aegean.gr

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Abstract

We investigate the solution of the quantal time-dependent linear oscillator from the viewpoint of the Lie point symmetries of its time-dependent Schrödinger equation. Four of the five nongeneric symmetries can be used for the construction of wavefunctions, two as creation symmetries and two as annihilation symmetries. The fifth nongeneric symmetry provides an eigenvalue for the Lie Bracket which maps a solution symmetry to itself. The general treatment indicates that care must be taken with the selection of solutions of the classical equation for the time-dependent linear oscillator and its third-order self-adjoint counterpart to obtain results which are consistent with the standard results for the autonomous harmonic oscillator.

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1. Introduction

In this paper we treat the quantal time-dependent linear oscillator from the viewpoint of the construction of its wavefunctions from the Lie symmetries of its time-dependent Schrödinger equation. In the process we uncover some critical features of the representation of the algebra of these Lie symmetries which have not been previously discussed. Indeed they were not obvious in recent treatments of the corresponding autonomous harmonic oscillator and related systems [1, 5] since the choice of representation for those discussions was made under the influence of the explicit functions present in the Lie symmetries obtained. Specifically we find that the representation of the algebra has an arbitrariness in the choice of the coefficient functions of the symmetries which must be constrained in order that the wavefunctions constructed using the symmetries are consistent with the asymptotic behaviour required. In our treatment here of the time-dependent linear oscillator the function $\omega(t)$ is arbitrary apart from some modest requirements of differentiability. In section 2 we report the Lie point symmetries of the Schrödinger equation for the time-dependent linear oscillator. Naturally the form of the Schrödinger equation must be the time-dependent form since separation of variables as in the case of the autonomous oscillator is not a proposition. In section 3 we demonstrate the construction of solutions of the time-dependent Schrödinger equation. Basis functions are obtained by considering similarity solutions invariant under one or other of the Lie point symmetries. Further solutions are generated by means of the action of symmetries on solution surfaces producing solutions, sometimes new, sometimes trivial. The actual implementation of this procedure is effected through a specific Lie Bracket. In this section we do not take into account the requirement that the wavefunction vanish at spatial infinity. In section 4 we see that an appropriate choice of the representation of the Lie point symmetries enables us to construct a consistent set of creation and annihilation operators for the time-dependent linear oscillator and that this set leads to wavefunctions with the required behaviour at spatial infinity. Finally in section 5 we present our concluding observations.

2. The Lie point symmetries

The time-dependent Schrödinger equation for the one-dimensional time-dependent linear oscillator is

$$2\mathrm{i}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \omega^2(t)x^2u = 0, \tag{2.1}$$

where t is the time variable, x is the space variable and $\omega^2(t)$ is the time-dependent 'frequency'. Apart from a requirement of differentiability there is no restriction upon the function $\omega(t)$ as far as the investigation of symmetry is concerned. However, when one considers the usual physical applications, the function $\omega(t)$ would be a positive real function.

Equation (2.1) possesses the symmetries

$$\begin{aligned}
\Gamma_1 &= \nu_1\partial_t + \tfrac{1}{2}x\dot{\nu}_1\partial_x + \tfrac{1}{4}(\mathrm{i}x^2\ddot{\nu}_1 - \dot{\nu}_1)u\partial_u \\
\Gamma_2 &= \nu_2\partial_t + \tfrac{1}{2}x\dot{\nu}_2\partial_x + \tfrac{1}{4}(\mathrm{i}x^2\ddot{\nu}_2 - \dot{\nu}_2)u\partial_u \\
\Gamma_3 &= \nu_3\partial_t + \tfrac{1}{2}x\dot{\nu}_3\partial_x + \tfrac{1}{4}(\mathrm{i}x^2\ddot{\nu}_3 - \dot{\nu}_3)u\partial_u \\
\Gamma_4 &= \mathrm{i}\sigma_1\partial_x - \dot{\sigma}_1xu\partial_u \\
\Gamma_5 &= \mathrm{i}\sigma_2\partial_x - \dot{\sigma}_2xu\partial_u \\
\Gamma_6 &= u\partial_u \\
\Gamma_7 &= f(t,x)\partial_u,
\end{aligned} \tag{2.2}$$

where $f(t,x)$ is any solution of (2.1), $\nu_i, i = 1, 2, 3$, are any three linearly independent solutions of

$$\dddot{\nu} + 4\omega^2\dot{\nu} + 4\omega\dot{\omega}\nu = 0 \tag{2.3}$$

and $\sigma_i, i = 1, 2$, are any two linearly independent solutions of

$$\ddot{\sigma} + \omega^2\sigma = 0. \tag{2.4}$$

Equation (2.3) is integrated to the Ermakov–Pinney equation by means of the integrating factor ν and the change of dependent variable $\nu = \rho^2$. Specifically we have

$$
\begin{aligned}
&\nu\,\dddot{\nu} + 4\omega^2\nu\dot{\nu} + 4\omega\dot{\omega}\nu^2 = 0\\
&\nu\ddot{\nu} - \frac{1}{2}\dot{\nu}^2 + 2\omega^2\nu^2 = 2K\\
&\ddot{\rho} + \omega^2\rho = \frac{K}{\rho^3},
\end{aligned}
\tag{2.5}
$$

where K is the constant of integration. Equation (2.5) is the Ermakov–Pinney equation. In 1950 the recently late Edmund Pinney [6] provided the solution in terms of the linearly independent solutions of the Newtonian equation of the classical time-dependent linear oscillator given by

$$\ddot{x} + \omega^2(t)x^2 = 0. \tag{2.6}$$

If a solution set of (2.6) is $\{\eta(t), \zeta(t)\}$, the solution of (2.5) is

$$\rho(t) = \sqrt{A\eta^2 + 2B\eta\zeta + C\zeta^2}, \tag{2.7}$$

where the three constants satisfy the relationship $B^2 - AC = W^2/(4K)$ and W is the Wronskian, namely $\eta\dot{\zeta} - \dot{\eta}\zeta$, of the solution set of (2.6) and is a constant. From this relation we may establish expressions for the linearly independent solutions of (2.3) in terms of those of (2.6). We take

$$\nu_1 = \eta^2 \tag{2.8}$$

$$\nu_2 = \eta\zeta \tag{2.9}$$

$$\nu_3 = \zeta^2. \tag{2.10}$$

We observe that this choice is arbitrary, although convenient for our purpose, and any other choice could be made. We emphasize through our choice of the symbols η and ζ for two linearly independent solutions of (2.6) that the solution set of (2.5) need not be constructed from the specific solution set of (2.6) used in the expressions for the coefficient functions of Γ_4 and Γ_5.

3. Construction of the wavefunctions

In the standard way we use the symmetries to construct solutions of the Schrödinger equation (2.1). Motivated by earlier experience with time-independent problems [1, 5] we select one of the solution symmetries, say Γ_4. The invariants of Γ_4 are obtained from the associated Lagrange's system [3]

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\mathrm{i}\sigma_1} = \frac{\mathrm{d}u}{-\dot{\sigma}_1 x u} \tag{3.1}$$

and are t and $u\exp\left[-\frac{1}{2}\mathrm{i}\dot{\sigma}_1 x^2/\sigma_1\right]$. We assume a similarity solution of the form

$$u = \exp\left[\frac{1}{2}\mathrm{i}\frac{\dot{\sigma}_1}{\sigma_1}x^2\right] f(t) \tag{3.2}$$

and substitute it into (2.1) to determine the form of the function $f(t)$. We find that $f(t) = \sigma_1^{-1/2}$ so that the basic solution is

$$u(t,x) = \sigma_1^{-\frac{1}{2}}\exp\left[\frac{1}{2}\mathrm{i}\frac{\dot{\sigma}_1}{\sigma_1}x^2\right]. \tag{3.3}$$

Closed-form solutions of (2.4) are not common. Some have been given in the paper of Eliezer and Grey [2] and we tabulate them here. We note that Eliezer and Grey give just $\omega(t)$ and $\rho^2(t)$. We include $\sigma_1(t)$ and $\sigma_2(t)$. For these we have made a particular choice of basis.

$$
\begin{array}{llll}
\omega(t) & \rho^2(t) & \sigma_1(t) & \sigma_2(t) \\
a/t & \alpha t & t^{(\alpha+2\mathrm{i})/2\alpha} & t^{(\alpha-2\mathrm{i})/2\alpha} \\
\frac{1}{2}/t & t[A(\log t+B)^2+A^{-1}] & t^{1/2} & t^{1/2}\log t \\
a/t^2 & \beta t & \beta t\exp\left[\dfrac{\mathrm{i}}{\beta^2 t}\right] & \beta t\exp\left[-\dfrac{\mathrm{i}}{\beta^2 t}\right] \\
at^k & t^{1/2}\left[J_\gamma^2(\tau)+Y_\gamma^2(\tau)\right] & J_\gamma(\tau) & Y_\gamma(\tau),
\end{array}
\tag{3.4}
$$

where $\alpha = \left(a^2 - \frac{1}{4}\right)^{-1/2}$ if $a \neq \frac{1}{2}$, $\beta = a^{-1/2}$, $\gamma = [2(k+1)]^{-1}$, $\tau = -2\gamma a t^{k+1}$ and J_γ and Y_γ are Bessel functions.

We observe the appearance of the basic solution for these different functions $\omega(t)$. For the first we have

$$u = (t^{(\alpha\pm\mathrm{i})/\alpha})^{-\frac{1}{2}} \exp\left[\frac{1}{2}\mathrm{i}\left(\frac{\alpha \pm 2\mathrm{i}}{2\alpha}\right)\frac{x^2}{t}\right]. \tag{3.5}$$

For the upper sign $u(t,x)$ has the correct behaviour at spatial infinity, but is otherwise unsatisfactory due to the imaginary exponent on t. In the case of the second the basic solutions are

$$u = t^{-1/4}\exp\left[\frac{\mathrm{i}}{4}\frac{x^2}{t}\right] \tag{3.6}$$

and

$$u = (t^{1/2}\log t)^{-1/2}\exp\left[\frac{\mathrm{i}}{4}\frac{x^2}{t}(1+\log t)\right]. \tag{3.7}$$

Neither of these solutions satisfies the physical requirements of the problem.

For the third we have

$$u = \left(\beta t\exp\left[\pm\frac{\mathrm{i}}{\beta^2 t}\right]\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\left(\mathrm{i}\frac{x^2}{t} \mp \left(\frac{x}{\beta t}\right)^2\right)\right]. \tag{3.8}$$

The upper sign gives acceptable behaviour at spatial infinity. In the case of the fourth solution provided by Eliezer and Grey we have

$$u = (Z_\gamma(t))^{-1/2}\exp\left[\frac{\mathrm{i}}{2}\frac{\dot{Z}_\gamma(t)}{Z_\gamma(t)}x^2\right], \tag{3.9}$$

where $Z_\gamma = J_\gamma$ or Y_γ. Not only do we have unsatisfactory behaviour at spatial infinity but there are also problems for finite time in the case of $J_\gamma(t)$ due to its zeros.

We see that wavefunctions based upon solution of (2.4) do not have good behaviour. In general, given an $\omega(t)$ and a solution $\rho(t)$ of the Ermakov–Pinney equation

$$\ddot{\rho} + \omega^2\rho = \rho^{-3}, \tag{3.10}$$

we have

$$\sigma = \rho\exp[\pm\mathrm{i}T], \qquad T = \int \rho^{-2}(t)\,\mathrm{d}t \tag{3.11}$$

which leads to the basic solution being

$$u = (\rho\exp[\pm\mathrm{i}T])^{-\frac{1}{2}}\exp\left[\frac{1}{2}\left(\mathrm{i}\frac{\dot{\rho}}{\rho} \mp \frac{1}{\rho^2}\right)x^2\right]. \tag{3.12}$$

The upper sign gives the correct behaviour at spatial infinity. However, for nonconstant $\rho(t)$ the wavefunction can oscillate wildly.

In a similar fashion we may use the other symmetries to generate solutions, but for the moment we concentrate on the solution (3.3) and use it to generate further solutions. There are two equivalent ways to generate further solutions. One is the use of the concept of a solution surface [1, 5] which is constructed as $\Sigma = u^{-1}u_n(t,x)$ and a new solution surface obtained by the action of one of the symmetries. For a general symmetry, $\Gamma = \tau\partial_t + \xi\partial_x + \upsilon u\partial_u$, the new solution surface is given as

$$\begin{aligned}\Sigma' &= \Gamma\{u^{-1}u_n(t,x)\}\\ &= u^{-1}\left[\tau\frac{\partial u_n}{\partial t} + \xi\frac{\partial u_n}{\partial x} - \upsilon\frac{\partial u_n}{\partial u}\right]\end{aligned} \tag{3.13}$$

and the new solution is given by the expression within the crochets. The other method is to make use of the property that the Lie Bracket of any of Γ_1–Γ_6 with one of Γ_7 gives another member of the infinite set of symmetries Γ_7. With the same symmetry as above the Lie Bracket is

$$\begin{aligned}[\Gamma,\Gamma_7]_{\mathrm{LB}} &= [\tau\partial_t + \xi\partial_x + \upsilon u\partial_u, f(t,x)\partial_u]_{\mathrm{LB}}\\ &= \left[\tau\frac{\partial f}{\partial t} + \xi\frac{\partial f}{\partial x} - \upsilon\frac{\partial f}{\partial u}\right]\partial_u\end{aligned} \tag{3.14}$$

and the expression within the crochets is a solution, possibly trivial, of (2.1).

The standard method to generate new solutions is to use the other solution symmetry, in this case $\Gamma_5 = \mathrm{i}\sigma_2\partial_x - \dot\sigma_2 xu\partial_u$. Successive applications give the solutions

$$u_1 = W_\sigma x\frac{u_0}{\sigma_1} \tag{3.15}$$

$$u_2 = W_\sigma\{\mathrm{i}\sigma_1\sigma_2 + W_\sigma x^2\}\frac{u_0}{\sigma_1^2} \tag{3.16}$$

$$u_3 = W_\sigma^2\{3\mathrm{i}x\sigma_1\sigma_2 + W_\sigma x^3\}\frac{u_0}{\sigma_1^3} \tag{3.17}$$

$$\vdots$$

We note that the action of Γ_4 is to lower the solutions. Thus we have the reductions $u_0 \longrightarrow 0$, $u_1 \longrightarrow \mathrm{i}W_\sigma u_0$, $u_2 \longrightarrow 2\mathrm{i}W_\sigma u_1$, $u_3 \longrightarrow 3\mathrm{i}W_\sigma u_2$ etc.

We consider now the actions of the $sl(2,R)$ symmetries on u_0. We recall the relationships between the solutions of (2.3) and (2.4). We have $\nu_1 = \eta^2$, $\nu_2 = \eta\zeta$ and $\nu_3 = \zeta^2$ with $\eta = \alpha_1\sigma_1 + \alpha_2\sigma_2$ and $\zeta = \beta_1\sigma_1 + \beta_2\sigma_2$ and obtain

$$\begin{array}{lll}\Gamma_1 \quad u_0 & \longrightarrow & \frac{1}{2}\alpha_2(\alpha_1 W_\sigma u_0 - \mathrm{i}\alpha_2 u_2)\\ \Gamma_2 \quad u_0 & \longrightarrow & \frac{1}{4}(\alpha_1\beta_2 + \alpha_2\beta_1)W_\sigma u_0 - \frac{1}{2}\mathrm{i}\alpha_2\beta_2 u_2\\ \Gamma_3 \quad u_0 & \longrightarrow & \frac{1}{2}\beta_2(\beta_1 W_\sigma u_0 - \mathrm{i}\beta_2 u_2).\end{array} \tag{3.18}$$

In these relationships we note that not one of Γ_1, Γ_2 or Γ_3 acts as a pure ladder operator. If we make the particular identifications

$$\begin{array}{lll}\nu_1 = \eta^2 = \sigma_1^2 & \alpha_1 = 1, & \alpha_2 = 0\\ \nu_2 = \eta\zeta = \sigma_1\sigma_2 & \beta_2 = 0, & \alpha_2 = 0\\ \nu_3 = \zeta^2 = \sigma_2^2 & \beta_1 = 0, & \beta_2 = 1,\end{array} \tag{3.19}$$

the results listed in (3.18) become

$$\begin{array}{lll}\Gamma_1 & u_0 & \longrightarrow & 0 \\ \Gamma_2 & u_0 & \longrightarrow & \frac{1}{4}W_\sigma u_0 \\ \Gamma_3 & u_0 & \longrightarrow & -\frac{1}{2}\mathrm{i}\beta_2^2 u_2\end{array} \tag{3.20}$$

so that Γ_1 plays the role of an annihilation operator and Γ_3 the role of a creation operator. In the case of Γ_3 there is a jump of two solutions instead of the single jump produced by Γ_5. Normally Γ_1 produces a drop of two states, but, since we were acting on the ground state, only the trivial solution is possible. In the action of Γ_2 we have an interesting result. The symmetry maps the solution to itself. This is the generalization of what happens with the time-independent oscillator for which the symmetry corresponding to Γ_2 is simply ∂_t. For that case the symmetry gives the energy. In the interpretation of the action of the symmetries on solution surfaces Γ_2 maps the surface to itself.

Equally we could apply the above analysis to solutions obtained commencing with Γ_5. Essentially, the same results are obtained.

We consider now the solution generated by the symmetry Γ_1. The invariants are obtained from the associated Lagrange's system

$$\frac{\mathrm{d}t}{\nu_1} = \frac{\mathrm{d}x}{\frac{1}{2}x\dot{\nu}_1} = \frac{\mathrm{d}u}{\frac{1}{4}(\mathrm{i}x^2\ddot{\nu}_1 - \dot{\nu}_1)u} \tag{3.21}$$

and are

$$v = \frac{x}{\nu_1^{\frac{1}{2}}} \qquad \text{and} \qquad w = u\nu_1^{\frac{1}{4}} \exp\left[-\frac{\mathrm{i}}{4}\frac{x^2\dot{\nu}_1}{\nu_1}\right] \tag{3.22}$$

so that we may write the similarity solution in the form

$$u = \nu_1^{-\frac{1}{4}} \exp\left[\frac{\mathrm{i}}{4}\frac{x^2\dot{\nu}_1}{\nu_1}\right] f\left(\frac{x}{\nu_1^{\frac{1}{2}}}\right). \tag{3.23}$$

When (3.23) is substituted into the Schrödinger equation, (2.1), the equation for f reduces to

$$f'' = 0 \quad \Longrightarrow \quad f = A_0 + A_1 v \tag{3.24}$$

and corresponding to the two constants of integration we obtain the two solutions

$$u_{1,0} = \frac{1}{\nu_1^{1/4}} \exp\left[\frac{\mathrm{i}}{4}x^2\frac{\dot{\nu}_1}{\nu_1}\right] = \frac{1}{\eta^{1/2}} \exp\left[\frac{\mathrm{i}}{2}x^2\frac{\dot{\eta}_1}{\eta_1}\right] \tag{3.25}$$

$$u_{1,1} = \frac{x}{\nu_1^{3/4}} \exp\left[\frac{\mathrm{i}}{4}x^2\frac{\dot{\nu}_1}{\nu_1}\right] = \frac{x}{\eta^{3/2}} \exp\left[\frac{\mathrm{i}}{2}x^2\frac{\dot{\eta}_1}{\eta_1}\right]. \tag{3.26}$$

Note that these two solutions correspond to the u_0 and u_1 above in terms of structure, but they are not the same since $\eta = \alpha_1\sigma_1 + \alpha_2\sigma_2$. Coincidence is established only if $\alpha_1 = 1$ and $\alpha_2 = 0$. In the case of Γ_3 the same situation applies with the exception that η is replaced by $\zeta = \beta_1\sigma_1 + \beta_2\sigma_2$.

If we apply the solution symmetries, Γ_4 and Γ_5, to the solution surfaces corresponding to (3.25) and (3.26), we obtain

$$\begin{array}{llll}\Gamma_4: & u_{1,0} & \longrightarrow & \alpha_2 W_\sigma u_{1,1} \\ & u_{1,1} & \longrightarrow & \left[\dfrac{\mathrm{i}\sigma_1}{\eta} - \dfrac{\alpha_2 W_\sigma x^2}{\eta^2}\right]u_{1,0} \\ \Gamma_5: & u_{1,0} & \longrightarrow & \alpha_1 W_\sigma u_{1,1} \\ & u_{1,1} & \longrightarrow & \left[\dfrac{\mathrm{i}\sigma_2}{\eta} + \dfrac{\alpha_1 W_\sigma x^2}{\eta^2}\right]u_{1,0}.\end{array} \tag{3.27}$$

We note that the action on $u_{1,0}$ is closed, but the action on $u_{1,1}$ is not closed unless η is identified with σ_1, equally σ_2.

The middle symmetry, Γ_2, is different—a not uncommon observation with this member of $sl(2, R)$—even though the equation has the same form as (3.23). Now we have that $f(v)$, where $v = x/(\eta\zeta)^{1/2}$, satisfies

$$f'' - \tfrac{1}{4}W^2v^2f = 0 \tag{3.28}$$

which is Whittaker's equation, also known as Weber's equation since it was originally investigated by Hermite [3, p 159].

4. Behaviourally correct wavefunctions

The solutions to the time-dependent Schrödinger equation constructed in the previous section do not have the suggestion of proper behaviour at spatial infinity since the term $\exp[\mathrm{i}\dot{\sigma}x^2/\sigma]$, be it σ_1 or σ_2, does not have the appearance of a function declining to 0 as $x \longrightarrow \infty$. We have provided a partial solution through the alternate representation of the solution. Here we expand upon this theme. A discussion [4] of the Lie point symmetries of the classical equation of motion of the time-dependent oscillator,

$$\ddot{x} + \omega^2(t)x = 0,$$

gives its eight Lie point symmetries as

$$\begin{aligned}
\Phi_1 &= \rho^2 \sin 2T\,\partial_t + x(\rho\dot{\rho}\sin 2T + \cos 2T)\partial_x \\
\Phi_2 &= \rho^2 \cos 2T\,\partial_t + x(\rho\dot{\rho}\cos 2T - \sin 2T)\partial_x \\
\Phi_3 &= \rho\sin T\,\partial_x \\
\Phi_4 &= \rho\cos T\,\partial_x \\
\Phi_5 &= \rho^2\partial_t + x\rho\dot{\rho}\partial_x \\
\Phi_6 &= x\partial_x \\
\Phi_7 &= \rho^{-1}x\sin T\,\partial_t + (\dot{\rho}\sin T + \rho^{-1}\cos T)x^2\partial_x \\
\Phi_8 &= \rho^{-1}x\cos T\,\partial_t + (\dot{\rho}\cos T - \rho^{-1}\sin T)x^2\partial_x,
\end{aligned} \tag{4.1}$$

where $\rho(t)$ is a solution of (3.10) and the 'new time',

$$T = \int \rho^{-2}(t)\,\mathrm{d}t.$$

We observe that there has been some selection in the choice of coefficient functions. We are at liberty to take different combinations if they suit our purpose. For the nonce we simply observe that Φ_3 and Φ_4 correspond to the 'solution' symmetries Γ_4 and Γ_5 in (2.2) but not necessarily respectively. In fact neither $\cos T$ nor $\sin T$ gives the requisite behaviour in $\mathrm{i}\dot{\sigma}x^2/\sigma$ required above since neither $-\mathrm{i}\cot T$ nor $\mathrm{i}\tan T$ makes the exponential term go to zero as $x \longrightarrow \infty$. However, the combinations $\cos T \pm \mathrm{i}\sin T = \exp[\pm\mathrm{i}T]$ do give the possibility for the required behaviour. Arbitrarily we take

$$\sigma_1 = \rho\exp[\mathrm{i}T] \qquad \text{and} \qquad \sigma_2 = \rho\exp[-\mathrm{i}T] \tag{4.2}$$

so that the solution based on Γ_4 has the correct behaviour at spatial infinity. This immediately casts Γ_5 as the creation operator while Γ_4 is the annihilation operator.

In fact we may use the symmetries of (4.1) to lead to the friendlier set of creation and annihilation symmetries for the Schrödinger equation of the time-dependent linear oscillator as

$$\Lambda_\pm = \exp[\pm \mathrm{i}T]\left\{\mathrm{i}\rho\partial_x - \left(\dot\rho \pm \frac{\mathrm{i}}{\rho}\right)xu\partial_u\right\} \tag{4.3}$$

the invariants of which are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\mathrm{i}\rho} = \frac{\mathrm{d}u}{-\left(\dot\rho \pm \dfrac{\mathrm{i}}{\rho}\right)xu} \tag{4.4}$$

to be t and

$$w = u\exp\left[\frac{1}{2}\left(-\frac{\mathrm{i}\dot\rho}{\rho} \pm \frac{1}{\rho^2}\right)x^2\right].$$

The similarity solution is taken to have the form

$$u = \exp\left[\frac{1}{2}\left(\frac{\mathrm{i}\dot\rho}{\rho} \mp \frac{1}{\rho^2}\right)x^2\right]h(t) \tag{4.5}$$

from which it is evident that the creation symmetry must be Λ_+. On substitution of (4.5) into (2.1) we obtain

$$2\mathrm{i}\dot h + \left\{\frac{\mathrm{i}\dot\rho}{\rho} - \frac{1}{\rho^2} + x\left(-\frac{\ddot\rho}{\rho} + \frac{1}{\rho^4} - \omega^2\right)\right\}h = 0 \tag{4.6}$$

which reduces to

$$\frac{\dot h}{h} = -\frac{1}{2}\left(\frac{\dot\rho}{\rho} + \frac{\mathrm{i}}{\rho^2}\right) \quad\Leftrightarrow\quad h = \rho^{-1/2}\exp\left[-\frac{1}{2}\mathrm{i}T\right] \tag{4.7}$$

when the Ermakov–Pinney equation is taken into account.

Thus we obtain

$$u_0 = \rho^{-1/2}\exp\left\{\frac{1}{2}\left[\left(\frac{\mathrm{i}\dot\rho}{\rho} - \frac{1}{\rho^2}\right)x^2 - \mathrm{i}T\right]\right\} \tag{4.8}$$

as the behaviourally correct ground state for the time-dependent linear oscillator. The Lie Bracket of Λ_- with Γ_7 of (2.2) gives

$$f_{\mathrm{new}} = \exp[-\mathrm{i}T]\left\{\mathrm{i}\rho\partial_x + \left(\dot\rho - \frac{\mathrm{i}}{\rho}\right)x\right\}f_{\mathrm{old}} \tag{4.9}$$

so that

$$u_n = \left\{\exp[-\mathrm{i}T]\left[\mathrm{i}\rho\partial_x + \left(\dot\rho - \frac{\mathrm{i}}{\rho}\right)x\right]\right\}^n u_0. \tag{4.10}$$

In particular we have for example

$$u_1 = -2\mathrm{i}\rho^{-1/2}\frac{x}{\rho}\exp\left\{\frac{1}{2}\left[\left(\frac{\mathrm{i}\dot\rho}{\rho} - \frac{1}{\rho^2}\right)x^2 - 3\mathrm{i}T\right]\right\} \tag{4.11}$$

$$u_2 = 2\rho^{-1/2}\left(1 - \frac{2x^2}{\rho^2}\right)\exp\left\{\frac{1}{2}\left[\left(\frac{\mathrm{i}\dot\rho}{\rho} - \frac{1}{\rho^2}\right)x^2 - \mathrm{i}T\right]\right\}. \tag{4.12}$$

Unlike the case of the autonomous harmonic oscillator for which $\mathrm{i}\partial_t$ is a symmetry and which gives the eigenvalue through

$$\mathrm{i}\partial_t u_n = E_n u_n \tag{4.13}$$

there is no symmetry listed in (2.2) with that obvious simplicity. However, we recall that the symmetry $\mathrm{i}\partial_t$ for the autonomous oscillator is an element of the algebra $sl(2, R)$ of its Schrödinger equation and that the choice of $\nu_2 = \sigma_1\sigma_2$ gave the very operator we seek, cf (3.26*b*). With the choice of σ_1 and σ_2 above we have

$$\Gamma_2 = \rho^2\partial_t + \rho\dot{\rho}x\partial_x + \tfrac{1}{2}[\mathrm{i}(\rho\ddot{\rho} + \dot{\rho}^2)x^2 - \rho\dot{\rho}]u\partial_u. \qquad (4.14)$$

The Lie Bracket of $\mathrm{i}\Gamma_2$ and Γ_7 of (2.2) with $f(t,x) = u_0$ is

$$[\mathrm{i}\Gamma_2, u_0(t,x)\partial_u]_{\mathrm{LB}} = \tfrac{1}{2}u_0\partial_u \qquad (4.15)$$

and in general we have

$$[\mathrm{i}\Gamma_2, u_n(t,x)\partial_u]_{\mathrm{LB}} = \left(n + \tfrac{1}{2}\right)u_n\partial_u \qquad (4.16)$$

which is an appropriate generalization of the result for the autonomous problem.

5. Observations

In treating the quantal time-dependent linear oscillator *ab initio* in terms of the Lie point symmetries of its time-dependent Schrödinger equation we have revealed various features which are not evident in standard treatments. The primary feature is that the time-dependent functions in the coefficients of the Lie point symmetries—σ_1 and σ_2 for the 'solution' symmetries and ν_1, ν_2 and ν_3 for the $sl(2, R)$ symmetries—need not be written in terms of the same basis. Normally this would be the case as is seen in the treatments of the corresponding autonomous systems [1, 5]. We emphasize that this is not because of necessity. One presumes that it is simply the mind's way to minimize the amount of memory required by recognizing patterns. We find that the mapping of solutions into solutions is not as tidy as previously reported in the autonomous case. In particular the pure ladder operators of that case are generally mixed operators here. Only the choice of a particular relationship between the νs and the σs restores the pure ladder property. We also find that the construction of wavefunctions does require some care in the selection of a suitable basis for σ_1 and σ_2. As can be seen from (3.4), the selection may not be so easy.

A final observation is the last result in section 4 in which we find that the eigenvalue of the Lie Bracket which maps a solution symmetry (of the time-dependent Schrödinger equation, i.e. Γ_7 of (2.2)) to itself is precisely that of the quantal time-independent harmonic oscillator (the setting of $K = 1$ in the Ermakov–Pinney equation, (2.5), is equivalent to taking $\Omega^2 = 1$ for the autonomous harmonic oscillator). This is a curious result. If one follows the transformation from time-dependent linear oscillator to time-independent harmonic oscillator outlined in section 1 in the context of the time-dependent Schrödinger equation, one obtains $n + \frac{1}{2}$ as the energy eigenvalue through the Lie Bracket relation

$$[\mathrm{i}\partial_t, u_n(t,x)\partial_u]_{\mathrm{LB}} = \mathrm{i}\frac{\partial u_n}{\partial t}\partial_u = \left(n + \frac{1}{2}\right)u_n\partial_u, \qquad (5.1)$$

where u_n is now the solution of the Schrödinger equation for the autonomous harmonic oscillator. However, this is the eigenvalue for the wavefunction of the Lewis invariant and not the wavefunction of the time-dependent linear oscillator.

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